UNITED STATES
NAVAL POSTGRADUATE SCHOOL

MUTUAL IMPEDANCE OF STACKED RHOMBIC ANTENNAS

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JESSE GERALD CHANEY
PROFESSOR OF ELECTRONICS

A REPORT
TO THE NAVY DEPARTMENT
BUREAU OF SHIPS
UPON AN INVESTIGATION CONDUCTED UNDER
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MUTUAL IMPEDANCE OF STACKED RHOMBIC ANTENNAS

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ABSTRACT

Unattenuated travelling waves are postulated as a first approximation to the current along two identical rhombic antennas stacked such that their axes are parallel and normal to the line joining their driving points. Their mutual radiation impedance is then found by the generalized circuit method. In order that the impedance may be separated into its resistive and reactive components, new associated sine integral and cosine integral functions are defined. A form of the new functions is given from which their tabulation should readily be possible. The mutual impedance formula is reducible to the formula for the radiation impedance of a single rhombic antenna.

INTEGRALS REQUIRED

The mutual impedance[1] of two rhombic antennas is given[2] by

$$jkZ_{21}/30 = \oint_1 \oint_2 e(r_{21})\left[\frac{\partial^2}{\partial s_1 \partial s_2} - k^2 \cos\theta(s_1, s_2)\right] g(ks_1, ks_2)\, ds_2\, ds_1 \qquad (1)$$

in which

$k = 2\pi/\lambda$

$g(ks_1, ks_2) = Re[f_1(P_1)^* f_2(P_2)]$

$\cos\theta(s_1, s_2) = d\bar{r}_1 \cdot d\bar{r}_2$

$e(r_{21}) = r_{21}^{-1} \exp(-jkr_{21})$

with

$f_i(P_i)$ = current distribution function

s_i = arc length coordinate

$\theta(s_1, s_2)$ = angle between path directions

r_{21} = distance between positions on different antennas

$*$ = complex conjugate to be taken

Re = real part to be taken

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1. J. G. Chaney, 'A critical study of the circuit concept', J. Appl.Phys. 22, 12, 1429, (1951)

2. J. G. Chaney, 'Simplification for mutual impedance of certain antennas', U. S. Naval Postgrad. School Tech Rpt. no. 6, Nov., 1952

Equation (1) also gives the external impedance of a single rhombic antenna if one path is taken along the axis of the wire and the other path is taken along the periphery of the wire.

In the case of a rhombic antenna in free space, after postulating an unattenuated travelling wave for a first approximation to the current distribution, equation (1) was integrated[3] for the radiation impedance of the terminated rhombic antenna. Due to symmetries involved and to equations of constraint satisfied by the current functions, the final integrations to be performed were reduced to three in number, the integrands being exponentials of the Murray[4] type.

Now, for two identical rhombic antennas stacked with a separation height h, the symmetries involved and the equations of constraint satisfied by the currents are identical with those encountered in the case of the single rhombic antenna. The only difference encountered in the mathematical set up of the problem is that the height h must be included in the formulas for the distances between two current moments, and that no approximations are required in the limits of integration. Hence, the mutual impedance Z_m of the stacked rhombic antennas is given by[3]

$$Z_m = -j\ 120k\ \sin^2\alpha\ (I_1 + I_2 - 2I_3) \tag{2}$$

in which

$$I_1 = \int_0^l\int_0^l exp[-jk(x_4+x_1+r_{41})]\,r_{41}^{-1}\,dx_4dx_1 \tag{3}$$

$$I_2 = \int_0^l\int_0^l exp[\ jk(x_4+x_1-r_{41})]\,r_{41}^{-1}\,dx_4dx_1 \tag{4}$$

$$I_3 = \int_0^l\int_0^l exp[-jk(x_2-x_1+r_{21})]\,r_{21}^{-1}\,dx_2dx_1 \tag{5}$$

$$r_{41} = (x_1{}^2+x_4{}^2+2x_1x_2cos2\alpha+h^2)^{\frac{1}{2}} \tag{6}$$

$$r_{21} = (x_1{}^2+x_2{}^2-2x_1x_2cos2\alpha+h^2)^{\frac{1}{2}} \tag{7}$$

with

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3. J. G. Chaney, 'Free space radiation impedance of rhombic antennas', U. S. Naval Postgrad. School Tech. Rpt. no. 4, May, 1952.
4. F. H. Murray, 'Mutual impedance of two skew antenna wires', Proc. I. R. E., 21,1,154, 1933.

2α = vertex angle at the driving point

l = length of a leg of the rhombic antenna

However, the integrals (3), (4), and (5) can not be completely integrated into the customary sine and cosine integrals of a real argument. For example, the following integrals remain,

$$\left(3\int_{kh}^{k(L-l)}-\int_{k(L+l)}^{k(N+2l)}+\int_{kh}^{k(L-l)}-\int_{k(L-l)}^{k(N-2L)}-2\int_{k(L-l)}^{kM}\right)\frac{te^{-jt}dt}{t^2+a^2} \tag{8}$$

in which

$$a = kh\tan\alpha$$
$$L = (h^2+l^2)^{\frac{1}{2}}$$
$$M = [h^2+(2l\sin\alpha)^2]^{\frac{1}{2}}$$
$$N = [h^2+(2l\cos\alpha)^2]^{\frac{1}{2}}$$

ASSOCIATED SINE AND COSINE INTEGRALS

For breaking the integrals (8) into their real and imaginary parts, it becomes desirable to define new generalized sine integral and cosine integral functions. These functions will be called the associated sine integral function and the associated cosine integral function, respectively. They will be defined as follows,

$$Sia(x,y) = \int_0^x \frac{t\sin t}{t^2+y^2}dt \tag{9}$$

$$Cia(x,y) = -\int_x^\infty \frac{t\cos t}{t^2+y^2}dt \tag{10}$$

from which it is obvious that

$$\begin{aligned} Sia(x,0) &= Six \\ Cia(x,0) &= Cix \end{aligned} \tag{11}$$

In writing formulas, it is convenient to shorten the notation to

$$\begin{aligned} Sia(x,y) &= Siy(x) \\ Cia(x,y) &= Ciy(x) \end{aligned} \tag{12}$$

Hence, in shorthand form

$$\begin{aligned} Siakx &= Sia(kx,a) \\ Ciakx &= Cia(kx,a) \\ Sibkx &= Sia(kx,b) \\ Cibkx &= Cia(kx,b),\ etc. \end{aligned} \tag{13}$$

Upon expanding *sint* and *cost* into their power series and rearranging the terms,

$$\frac{t\ sint}{t^2+y^2} = \frac{sint}{t} - \frac{y\ sinht}{t^2+y^2} + \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{(-1)^{\mu-1} t^{2\mu-2} y^{2\nu}}{(2m+2n-1)!} \tag{14}$$

$$\frac{t\ cost}{t^2+y^2} = \frac{t\ coshy}{t^2+y^2} - \frac{1-cost}{t} + \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{(-1)^{\mu} t^{2\mu-1} y^{2\nu}}{(2m+2n)!} \tag{15}$$

Integrating equation (14),

$$Sia(x,y) = Si\,x - sinhy\ tan^{-1}\frac{x}{y} + \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{(-1)^{\mu-1} x^{2\mu-1} y^{2\nu}}{(2m-1)!(2m+2n-1)!} \tag{16}$$

For integrating equation (15), it is preferable to break up the integral as follows,

$$Cia(x,y) = \int_0^x \frac{t\ cost}{t^2+y^2}dt - \int_0^{\infty} \frac{t\ cost}{t^2+y^2}dt \tag{17}$$

The infinite integral in equation (17) may be evaluated by adding a pair of integrals from Jahnke and Emde[5]. That is,

$$\begin{aligned} Cia(0,y) &= -\int_0^{\infty} \frac{t\ cost}{t^2+y^2}dt \\ &= \tfrac{1}{2}[e^{-y}Ei(y) + e^{y}Ei(-y)] \end{aligned} \tag{18}$$

with

$$\begin{aligned} Ei(x) &= \int_{-\infty}^{x} e^{t}/t\ dt \\ Ei(-x) &= -\int_{x}^{\infty} e^{-t}/t\ dt \end{aligned} \tag{19}$$

Parenthetically, it follows from equation (16) that

$$Sia(0,y) = 0 \tag{20}$$

Thus, substituting from equation (15) for the integrand in the first integral of equation (17), it follows that

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5. E. Jahnke and F. Emde, 'Tables of Functions with Formulae and Curves', Dover Publications, New York, 1945

$$Cia(x,y) = Cix - C - \ln x + \cosh y \ \ln[(x/y)^2+1]^{\frac{1}{2}} + \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{(-1)^{\mu}\ x^{2\mu}y^{2\nu}}{(2m)!\,(2m+2n)!} + \tfrac{1}{2}[e^{y}Ei(-y)+e^{-y}(Ei(y))] \tag{21}$$

with $C = 0.5772...$

It may be demonstrated by contour integration that

$$Sia(\infty,y) = \tfrac{1}{2}\pi e^{-y} \tag{22}$$

Also, it follows from the definition that

$$Cia(\infty,y) = 0 \tag{23}$$

Furthermore, from the definitions,

$$\begin{aligned} Sia(-x,\pm y) &= -\ Sia(x,y) \\ Cia(-x,\pm y) &= Cia(x,y) \end{aligned} \tag{24}$$

For both $|y|<<1$ and $|x|<<1$,

$$\begin{aligned} Sia(x,y) &= x - y\tan^{-1}x/y + O(x^3, xy^2) \\ Cia(x,y) &= C + \ln(x^2+y^2) + O(x^2, y^2) \end{aligned} \tag{25}$$

For $|x|<<1$,

$$\begin{aligned} Sia(x,y) &= 0 + O(x^3) \\ Cia(x,y) &= Cia(0,y)+O(x^2) \end{aligned} \tag{26}$$

and for $|y|<<1$,

$$\begin{aligned} Sia(x,y) &= Six - \tfrac{1}{2}\pi + O(y^2) \\ Cia(x,y) &= C + \ln(x^2+y^2)^{\frac{1}{2}} + O(x^2, y^2) \end{aligned} \tag{27}$$

For $0<y<x$,

$$|Sia(x,y)|\leq 2/y, \quad |Cia(x,y)|\leq 1/y \tag{28}$$

and for $0<x<y$,

$$|Sia(x,y)|\leq 1/y, \quad |Cia(x,y)|\leq 2/y \tag{29}$$

Thus,

$$\mathrm{Sia}(x,\infty) = Cia(x,\infty) = 0 \tag{30}$$

In terms of the hyperbolic sine integral function along with sine integral and cosine integral functions of a complex argument $z=x+jy$, the associated functions may be written in the form,

$$\mathrm{Sia}(x,y) = \cosh y\ Re(Siz) + \sinh y[Im(Ciz)-\pi/2] \qquad (31)$$

$$\mathrm{Cia}(x,y) = \cosh y\ Re(Ciz) - \sinh y[Im(Siz)+Shiy] \qquad (32)$$

in which,

$$Siz = \int_{0+jy}^{x+jy} \sin t/t\ dt \qquad (33)$$

$$Ciz = \int_{\infty+jy}^{x+jy} \cos t/t\ dt \qquad (34)$$

$$Shiy = \int_0^y \mathrm{Sinh}\,t/t\ dt \qquad (35)$$

$$Rez = x, \quad Imz = y \qquad (36)$$

The infinite series form for equations (31) and (32) is expressible in terms of the functions $T_n(z)$ *defined*[6] *by*

$$T_n(z) = z^n/n\cdot n! \qquad (37)$$

Thus

$$\begin{aligned} Sia(x,y) &= \cosh y Re[T_1(z)-T_3(z)+T_5(z)-\ldots] \\ &\quad - \sinh y\{\tan^{-1}x/y + Im[T_2(z)-T_4(z)+T_6(z)-\ldots]\} \qquad (38) \end{aligned}$$

$$\begin{aligned} Cia(x,y) &= \cosh y\{C + \ln|z| - Re[T_2(z)-T_4(z)+T_6(z)-\ldots]\} \\ &\quad - \sinh y\{T_1(y)+T_3(y)+T_5(y)+\ldots \qquad (39) \\ &\qquad + Im[T_1(z)-T_3(z)+T_5(z)-\ldots]\} \end{aligned}$$

Equations (37) and (38) should be useful in the tabulation of the associated sine integral and cosine integral functions.

MUTUAL IMPEDANCE FORMULA

Upon carrying out the integrations and setting

$$a = kh\ \tan\alpha$$

the mutual impedance of two stacked identical rhombic antennas separated by a height h may be written as

6. W.P.A., New York, 'Tables of Sine, Cosine, and Exponential Integrals', 1949.

$$
\begin{aligned}
Z_m/120 = {} & 4[Ci(kh)-Cia(kh)] + 2[2Ciak(L-l)-Cik(L-l)] + 2[2Ciak(L+l)-Cik(L+l)] \\
& - 2Cia(kM) - Ciak(2l+N) - Ciak(2l-N) \\
& + \cos(2kl\sin^2\alpha)[Cik(N+2l\cos^2\alpha) + Cik(N-2l\cos^2\alpha) + Cik(M+2l\sin^2\alpha) \\
& \quad + Cik(M-2l\sin^2\alpha) - 2Cik(L+l\cos 2\alpha) - 2Cik(L-l\cos 2\alpha)] \\
& - \sin(2kl\sin^2\alpha)[Sik(N+2l\cos^2\alpha) - Sik(N-2l\cos^2\alpha) - Sik(M+2l\sin^2\alpha) \\
& \quad + Sik(M-2l\sin^2\alpha) - 2Sik(L+l\cos 2\alpha) + 2Sik(L-l\cos 2\alpha)] \\
& + j\{\, 4[Sia(kh)-Si(kh)] + Siak(N+2l) + Siak(N-2l) + 2Sia(kM) \qquad (40) \\
& \quad - 2[2Siak(L+l)-Sik(L+l)] - 2[2Siak(L-l)-Sik(L-l)] \\
& - \cos(2kl\sin^2\alpha)[Sik(N+2l\cos^2\alpha) + Sik(N-2l\cos^2\alpha) + Sik(M-2l\sin^2\alpha) \\
& \quad + Sik(M+2l\sin^2\alpha) - 2Sik(L+l\cos 2\alpha) - 2Sik(L-l\cos 2\alpha)] \\
& - \sin(2kl\sin^2\alpha)[Cik(N+2l\cos^2\alpha) - Cik(N-2l\cos^2\alpha) + Cik(M-2l\sin^2\alpha) \\
& \quad - Cik(M+2l\sin^2\alpha) - 2Cik(L+l\cos 2\alpha) + 2Cik(L-l\cos 2\alpha)] \,\}
\end{aligned}
$$

with

$$L = [h^2+l^2]^{\frac{1}{2}}, \quad M = [h^2+(2l\sin\alpha)^2]^{\frac{1}{2}}, \quad N = [h^2+(2l\cos\alpha)^2]^{\frac{1}{2}}$$

$$Si(x) = \int_0^x \sin t/t \, dt, \qquad Ci(x) = \int_\infty^x \cos t/t \, dt$$

$$Sia(x) = \int_0^x \frac{t \sin t}{t^2+a^2} dt, \qquad Cia(x) = \int_\infty^x \frac{t \cos t}{t^2+a^2} dt, \quad a = kh\tan\alpha$$

Formula (40) readily reduces to the formula for the self impedance of a rhombic antenna if h is permitted to vanish. Since both h and a vanish, equations (25) should be used for evaluating the limit of the expressions in the first line of formula (40).

APPENDIX A

Given

$$Sia(x) = \int_0^x \frac{t\ sint}{t^2+a^2}\,dt \qquad (1)$$

Breaking into partial fractions and transforming,

$$\begin{aligned} 2Sia(x) &= \int_0^x [1/(t+ja) + 1/(t-ja)]sint\ dt \\ &= \int_{0+ja}^{x+ja} sin(z-ja)/z\ dz + \int_{0-ja}^{x-ja} sin(z+ja)/z\ dz \\ &= cosha\int_{0+ja}^{x+ja} sinz/z\ dz - jsinha\int_{0+ja}^{x+ja} cosz/z\ dz \\ &+ cosha\int_{0-ja}^{x-ja} sinz/z\ dz + jsinha\int_{0-ja}^{x-ja} cosz/z\ dz \end{aligned} \qquad (2)$$

Considering the limits involved in the definition of the cosine integral, equation (2) may be written

$$\begin{aligned} 2Sia(x) &= cosha[Si(x+ja)+Si(x-ja)]-jsinha[Ci(x+ja)-Ci(x-ja)] \\ &+ jsinha\left[\int_{0-ja}^{\infty-ja} - \int_{0+ja}^{\infty+ja}\right] cosz/z\ dz \end{aligned} \qquad (3)$$

The remaining integral may be transformed back to the real form,

$$\begin{aligned} -j\left[\int_{0+ja}^{\infty+ja} - \int_{0-ja}^{\infty-ja}\right] cosz/z\ dz &= -j\int_0^\infty \left[\frac{cos(x+ja)}{x+ja} - \frac{cos(x-ja)}{x-ja}\right]dx \\ &= -2Sinha\int_0^\infty \frac{xsinx}{x^2+a^2}dx - 2acosha\int_0^\infty \frac{cosx}{x^2+a^2}dx \\ &= -2sinha(\pi e^{-a}/2) - 2acosha(\pi e^{-a}/2a) = -\pi \end{aligned} \qquad (5)$$

the latter integrals being well known and suitable for evaluation by contour integration.

Substituting from equation (5) into equation (3),

$$Sia(x) = Re\ Si(x+ja) + Sinha[Im\ Ci(x+ja) - \pi/2] \qquad (6)$$

or

$$Sia(x,y) = Re\ Siz + sinhy\ Im(Ciz-\pi/2) \qquad (7)$$

APPENDIX B

Given

$$\mathrm{Cia}(x) = -\int_0^{\infty} \frac{t\cos t}{t^2+a^2} dt \qquad (1)$$

Breaking into partial fractions and transforming as before,

$$\begin{aligned} 2Cia(x) &= -\int_{x+ja}^{\infty+ja} \cos(z-ja)/z \, dz - \int_{x-ja}^{\infty-ja} \cos(z+ja)/z \, dz \\ &= -\cosh a\left[\int_{x+ja}^{\infty+ja} + \int_{x-ja}^{\infty-ja}\right] \cos z/z \, dz \\ &\quad -j\sinh a\left[\int_{x+ja}^{\infty+ja} - \int_{x-ja}^{\infty-ja}\right] \sin z/z \, dz \end{aligned} \qquad (2)$$

Breaking up the latter integral,

$$\begin{aligned} 2Cia(x) &= \cosh a[Ci(x+ja)+Ci(x-ja)] + j\sinh a[Si(x+ja)-Si(x-ja)] \\ &\quad - j\sinh a\left[\int_{0+ja}^{\infty+ja} - \int_{0-ja}^{\infty-ja}\right] \sin z/z \, dz \end{aligned} \qquad (3)$$

Again considering the latter integral,

$$\begin{aligned} -j\left[\int_{0+ja}^{\infty+ja} - \int_{0-ja}^{\infty-ja}\right] \sin z/z \, dz &= -j\int_0^{\infty}\left[\frac{\sin(x+ja)}{x+ja} - \frac{\sin(x-ja)}{x-ja}\right]dx \\ &= 2\sinh a\int_0^{\infty}\frac{x\cos x}{x^2+a^2} dx - 2a\cosh a\int_0^{\infty}\frac{\sin x}{x^2+a^2} dx \end{aligned} \qquad (4)$$

The resulting integrals in equation (4) may be taken from Jahnke and Emde[5] as

$$\int_0^{\infty} \frac{x\cos x}{x^2+a^2} dx = -\tfrac{1}{2}[e^{a}Ei(-a)+e^{-a}Ei(a)] \qquad (5)$$

$$\int_0^{\infty} \frac{\sin x}{x^2+a^2} dx = -\tfrac{1}{2}[e^{a}Ei(-a)-e^{-a}Ei(a)] \qquad (6)$$

Substituting from equations (5) and (6) into equation (4) and subsequently into equation (2),

$$Cia(x) = \cosh a \ \mathrm{Re}\ Ci(x+ja) - \sinh a\{\mathrm{Im}\ Si(x+ja) + \tfrac{1}{2}[Ei(a)-Ei(-a)]\} \qquad (7)$$

Now consider

$$\begin{aligned} Ei(x) &= \int_{-\infty}^{x} e^{y}/y \, dy \\ &= \left(\int_{-\infty}^{-x} + \int_{-x}^{0} + \int_{+0}^{x}\right) e^{y}/y \, dy \\ &= \int_0^x (e^{y}-e^{-y})/y \, dy - \int_x^{\infty} e^{-y}/y \, dy \\ &= 2\int_0^x \sinh y/y \, dy + Ei(-x) \end{aligned} \qquad (8)$$

Hence

$$Ei(x) - Ei(-x) = 2\ Shi x \qquad (9)$$

and thus

$$\mathrm{Cia}(x,y) = \cosh y \ \mathrm{Re}\ Ciz - \sinh y\{\mathrm{Im}\ Siz + Shi y\} \qquad (10)$$

APPENDIX C

Given

$$I_1 = \int_0^l \int_0^l r_{41}^{-1} \exp[-jk(x_1+x_4+r_{41})]\,dx_4\,dx_1 \tag{1}$$

with

$$r_{41} = \sqrt{x_1^2+x_4^2+2x_1x_4\cos 2\alpha+h^2}$$

$$r_{4l} = \sqrt{x_4^2+2lx_4\cos 2\alpha+l^2+h^2}$$

$$r_{40} = \sqrt{x_4^2+h^2}$$

Let

$$m_1 t = x_1+x_4\cos 2\alpha+r_{41}, \quad m_1 = \sqrt{h^2+(x_4\sin 2\alpha)^2} \tag{2}$$

Then it follows that

$$m_1 t_1 = x_4\cos 2\alpha + \sqrt{x_4^2+h^2} \tag{3a}$$

$$m_1 t_1]_l = l\cos 2\alpha + \sqrt{l^2+h^2} \tag{3b}$$

$$m_1 t_1]_0 = h \tag{3c}$$

$$m_1 t_2 = l + x_4\cos 2\alpha + r_{4l} \tag{4a}$$

$$m_1 t_2]_l = 2l\cos^2\alpha + \sqrt{(2l\cos\alpha)^2+h^2} \tag{4b}$$

$$m_1 t_2]_0 = l + \sqrt{l^2+h^2} \tag{4c}$$

and

$$I_1 = \int_0^l \int_{t_1}^{t_2} e^{-jk(m_1 t+2x_4\sin^2\alpha)}\,dt\,dx_4$$

$$= \int_0^l e^{-j2kx_4\sin^2\alpha} \int_{km_1t_1}^{km_1t_2} (e^{-jt}/t)\,dt\,dx_4 \tag{5}$$

Integrating equation (5) by parts,

$$-j2k\sin^2\alpha\, I_1 = \left(e^{-j2kx_4\sin^2\alpha}\{Ci(km_1t_2) - Ci\,k(m_1t_2) - j[Si(km_1t_2) - Si\,k(m_1t_1)]\}\right)_0^l$$

$$-\int_0^l e^{-j2kx_4\sin^2\alpha}\left[\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_4} - \frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_4}\right. \tag{6}$$

Designate the integral of equation (6) by A_{11} and substitute from equations (3a) and (4a),

$$A_{11} = -\int_0^l \frac{e^{-jk(l+x_4+r_{4l})}[x_4+(l+r_{4l})\cos 2a]}{r_{4l}(l+x_4\cos 2a+r_{4l})}dx_4$$

$$+\int_0^l \frac{e^{-jk(x_4+r_{40})}(x_4+r_{40}\cos 2a)}{r_{40}(x_4\cos 2a+r_{40})}dx_4. \qquad (7)$$

Let A_{12} represent the second integral of equation (7) and change the variable by letting

$$hu = r_{40}+x_4, \quad hu_1 = h, \quad hu_2 = l+\sqrt{h^2+l^2} \qquad (8)$$

then

$$\int_{u_1}^{u_2} \frac{e^{-jkhu}}{u} \frac{h(u^2-1)/2u + h(u^2+1)\cos 2a/2u}{h(u^2+1)/2u + h(u^2-1)\cos 2a/2u}du \qquad (9a)$$

$$= \int_{u_1}^{u_2} \frac{e^{-jkhu}(u^2 - \tan^2 a)}{u(u^2 + \tan^2 a)}du \qquad (9b)$$

Breaking into partial fractions and letting

$$t = khu ,$$

$$A_{12} = -\int_{khu_1}^{khu_2} e^{-jt}/t \; dt + 2\int_{khu_1}^{khu_2} \left(\frac{te^{-jt}}{t^2+(kh\tan a)^2}\right)dt \qquad (10)$$

$$= -Ci(khu_2)+Ci(khu_1)+j[Si(khu_2)-Si(khu_1)]$$

$$+ 2\int_{khu_1}^{khu_2} \frac{te^{-jt}}{t^2+(kh\tan a)^2}dt \qquad (11)$$

Now let A_{13} designate the first integral of equation (7) and change the variable by letting

$$m_2 y = r_{4l}+x_4+l\cos 2a , \qquad m_2 = \sqrt{h^2+(l\sin 2a)^2} \qquad (12)$$

$$m_2 y_2 = r_{ll}+2l\cos^2 a , \qquad r_{ll} = \sqrt{h^2+(2l\cos a)^2} \qquad (13)$$

$$m_2 y_1 = r_{0l}+l\cos 2a , \qquad r_{0l} = \sqrt{h^2+l^2} \qquad (13b)$$

and get

$$A_{13} = -e^{-j2kl\sin^2 a}\int_{y_1}^{y_2} \frac{e^{-jkm_2 y}[\frac{m_2}{2y}(y^2-1)+\frac{m_2}{2y}(y^2+1)\cos 2a]dy}{y[\frac{m_2}{2y}(y^2+1)+\frac{m_2}{2y}(y^2-1)\cos 2a+l\sin^2 2a} \qquad (14)$$

Letting

$$m_2 p = l\sin 2\alpha \tag{15}$$

equation (14) reduces to

$$A_{13} = -e^{-j2kl\sin^2\alpha}\int_{y_1}^{y_2} \frac{e^{-jkm_2 y}(y^2-\tan^2\alpha)}{y(y^2+2py\tan\alpha+\tan^2\alpha)}dy \tag{16a}$$

$$= e^{-j2kl\sin^2\alpha}\int_{y_1}^{y_2} e^{-jkm_2 y}\left[1/y - \frac{2(y+p\tan\alpha)}{y^2+2py\tan\alpha+\tan^2\alpha}\right]dy \tag{16b}$$

In the first term, let

$$t = km_2(y) \quad t_2 = km_2 y_2, \quad t_1 = km_2 y_1 \tag{17a}$$

and in the second term, let

$$t = km_2(y+\tan\alpha) \tag{17b}$$

$$t_2 = km_2 y_2 + 2kl\sin^2\alpha, \quad t_1 = km_2 y_1 + 2kl\sin^2\alpha \tag{17c}$$

and get

$$A_{13} = e^{-j2kl\sin^2\alpha}\{Ci(km_2 y_2) - Ci(km_2 y_1) - j[Si(km_2 y_2) - Si(km_2 y_1)]\}$$
$$- 2\int_{km_2 y_1+2kl\sin^2\alpha}^{km_2 y_2+2kl\sin^2\alpha} \frac{te^{-jt}\,dt}{t^2+(kh\tan\alpha)^2} \tag{18}$$

Substituting A_{13} and A_{12} into equation (7), A_{11} into equation (6), and substituting for the limits from equations (3b,c), (4b,c), (8), (17a,c), upon letting

$$L = \sqrt{h^2+l^2}, \quad N = \sqrt{h^2+(l\cos 2\alpha)^2} \tag{19}$$

equation (6) becomes

$$-jk\sin^2\alpha\; I_1 = e^{-j2kl\sin^2\alpha}\{Cik(N+2l\cos^2\alpha) - Cik(L+\cos 2\alpha) - j[Sik(N+2l\cos^2\alpha)$$
$$-Sik(N-2l\cos^2\alpha)]\} - Cik(L+l) + Cikh + j[Sik(L+l) - Sikh]$$
$$+ \left[\int_{kh}^{k(L+l)} - \int_{k(L+l)}^{k(N+2l)}\right] \frac{te^{-t}}{t^2+(kh\tan\alpha)^2}dt \tag{20}$$

For

$$I_2 = \int_0^l\int_0^l r_{41}^{-1}\exp[-jk(r_{41}-x_1-x_4)]\,dx_4\,dx_1 \tag{21}$$

let

$$m_1 t = r_{41}-x_1-x_4\cos 2\alpha, \qquad m_1 = \sqrt{h^2+(x_4\sin 2\alpha)^2} \tag{22}$$

$$hu = r_{40}-x_4 \tag{23}$$

$$m_2 y = r_{4l}-(x_4+l\cos 2\alpha), \qquad m_2 = \sqrt{h^2+(l\sin 2\alpha)^2} \tag{24}$$

and proceed exactly as in I_1. Then

$$-jk\sin^2 I_2 = e^{j2kl\sin^2\alpha}\{Cik(N-2l\cos^2\alpha)-Cik(L-l\cos 2\alpha) - j[Sik(N-2l\cos^2\alpha)$$
$$-Sik(L-l\cos 2\alpha)]\} -Cik(L-l)+Cikh +j[Sik(L-l)-Sikh]$$
$$+[\int_{kh}^{k(L-l)} -\int_{k(L-l)}^{k(N-2l)}]\frac{te^{-t}}{t^2+(kh\tan\alpha)^2}dt \tag{25}$$

Likewise, for

$$I_3 = \int_0^l\int_0^l r_{21}^{-1}\exp[-jk(r_{21}-x_1+x_2)]dx_2dx_1 \tag{26}$$

with

$$r_{21} = \sqrt{x_1^2 + x_2^2 - 2x_1x_2\cos 2\alpha + h^2}$$

let

$$m_1 t = r_{21}-x_1+x_2\cos 2\alpha, \quad m_1 = \sqrt{h^2+(x_2\sin 2\alpha)^2} \tag{27}$$

$$hu = r_{20}+x_2 \tag{28}$$

$$m_2 y = r_{2l}+x_2-l\cos 2\alpha, \quad m_2 = \sqrt{h^2+(l\sin 2\alpha)^2} \tag{29}$$

and get

$$j2k\sin^2\alpha\ I_3 = e^{-j2kl\sin^2\alpha}\{Cik(M-2l\sin^2\alpha)-Cik(L+\cos 2\alpha) - j[Sik(M-2l\sin^2\alpha)$$
$$-Sik(L+l\cos 2\alpha)]\} -Cik(L-l)+Cikh + j[Sik(L+l)-Sikh]$$
$$+e^{j2kl\sin^2\alpha}\{Cik(M+2l\sin^2\alpha)-Cik(L-l\cos 2\alpha) - j[Sik(M+2l\sin^2\alpha)$$
$$-Sik(L-l\cos 2\alpha)]\} \tag{30}$$
$$+2[\int_{kh}^{k(L+l)} -\int_{k(L-l)}^{kM}]\frac{te^{-t}}{t^2+(kh\tan\alpha)^2}dt$$

in which

$$M = \sqrt{h^2+(l\sin 2\alpha)^2},$$

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